Irreducible Representations of S_n

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Abstract

In this paper we prove by a geometrical method an extension of Ruch-Schonhofer theorem by finding an explicit basis for the intertwining space. Then we find all the irreducible representations of S_n as images of one of these operators in a natural representation.

Introduction

This paper achieves two goals. The first one is that we find geometrically how much is $i(IS_\alpha \uparrow S_n, AS_{\beta'} \uparrow S_n)$ via the study of intertwining operators. It now becomes geometrically clear why $i(IS_\alpha \uparrow S_n, AS_{\alpha'} \uparrow S_n) = 1$, not as in the classical approach via characters. Second, we have a new method of finding the complete set of irreducible representations of S_n, as the image of an intertwinig operator between $IS_\alpha \uparrow S_n$ and $AS_{\alpha'} \uparrow S_n$. Maybe this method has a more general scope and could be applied to other groups like $GL_n(q)$.

Results

All unexplained notations are from [1]. The equivalence class of the α- tableau t modulo $H(t)$ being acted by the left is denoted by $\{t\}$ (H(t) defined in [1], pp 29). Let $\alpha \vdash n$ ([1], pp 16). Let $P_\alpha := \{\{t^\alpha\}\}$ (t^α in [1], pp 28). Let's think of $\ell(P_\alpha)$ (the space of all complex functions over P_α) as a vector space endowed with the natural action of S_n, this is, for every $\sigma \in S_n$, $\rho \in P_\alpha$, $f \in \ell(P_\alpha)$, we have $\sigma f(\rho) = f(\sigma^{-1}\rho)$. We will call this representation $I\ell(P_\alpha) \cong IS_\alpha \uparrow S_n := \mathrm{ind}_{S_\alpha}^{S_n}(\mathrm{Id})$ (because there is an element ρ of P such that Stab(ρ)=S_α) .

Let's think now of $\ell(P_\beta)$ as a vectorial space provided with the following action of S_n: for every $\sigma \in S_n$, $\rho_\beta \in P_\beta$, $f_\beta \in \ell(P_\beta)$, we have $\sigma f_\beta(\rho_\beta) = \mathrm{sgn} f_\beta(\sigma^{-1}\rho_\beta)$. We will call this representation $A\ell(P_\beta) \cong AS_\beta \uparrow S_n := \mathrm{ind}_{S_\beta}^{S_n}(\mathrm{sgn})$. Let $\phi : A\ell(P_\beta) \longrightarrow I\ell(P_\alpha)$ be an intertwining operator. Then for every $f_\beta \in A\ell(P_\beta)$, we have $\phi(f_\beta)(\rho) := \sum_{\rho_\beta \in P_\beta} \kappa(\rho_\beta, \rho) f_\beta(\rho_\beta)$.

Be $\rho \in P_\alpha$, $\pi \in P_\beta$. If $\tau \in S_n$ we have:

$$(\tau\phi(f_\beta))(\tau\rho) = (\phi(\tau f_\beta))(\tau\rho)$$

Then $\phi(f_\beta)(\rho) = \phi(f_\beta)(\tau^{-1}\tau\rho) = \sum_{\rho_\beta \in P_\beta} \kappa(\rho_\beta, \tau\rho)(\tau f_\beta)(\rho_\beta)$.

$$\sum_{\rho_\beta \in P_\beta} \kappa(\rho_\beta,\rho) f_\beta(\rho_\beta) = \sum_{\rho_\beta \in P_\beta} \kappa(\rho_\beta,\tau\rho) sgn(\tau) f_\beta(\tau^{-1}\rho_\beta) = \sum_{\tau\rho_\beta \in P_\beta} \kappa(\tau\rho_\beta,\tau\rho) sgn(\tau) f_\beta(\rho_\beta)$$

As this is true for any $f_\beta \in \ell(P_\beta)$, let $f_\beta(\rho_\beta) = \begin{cases} 0 & \text{if} \quad \rho_\beta \neq \pi \\ 1 & \text{if} \quad \rho_\beta = \pi \end{cases}$

Then

$$\kappa(\pi,\rho) = \text{sgn}(\tau)\kappa(\tau\pi,\tau\rho) \tag{1}$$

Lemma 1 *Let $\rho \in P_\alpha$. If $\pi \in P_\beta$ is such that there exists a transposition $\tau \in S_n$ such that $\tau\rho = \rho$ and $\tau\pi = \pi$, then $\kappa(\pi,\rho) = 0$.*

Proof:

$$\kappa(\pi,\rho) = \text{sgn}(\tau)\kappa(\tau\pi,\tau\rho) = -\kappa(\pi,\rho) \tag{2}$$

□

Def: $\Omega_\beta(t^\alpha) := \{\pi \in P_\beta / \sigma\pi \neq \pi \ \text{ for every } \ \sigma \in H(\{t^\alpha\}) \setminus \{Id\}\}$

Clearly in this definition, $|\Omega_\beta(t^\alpha)|$ does not depend in t^α. It only depends in α and β.

Lemma 2 *If $\pi_1 \in \Omega_\beta(t^\alpha), H(t^\alpha)\pi_1 \subseteq \Omega_\beta(t^\alpha)$*

Proof: Let $\delta \in H(t^\alpha), \sigma \in H(t^\alpha) \setminus \{Id\}$. If $\sigma\delta\pi_1 = \delta\pi_1$, then $\delta^{-1}\sigma\delta\pi_1 = \pi_1$, but as $\delta^{-1}\sigma\delta \in H(t^\alpha)$, $\delta^{-1}\sigma\delta = Id$. Then $\sigma = Id$. It follows that $H(t^\alpha)\pi_1 \subset \Omega_\beta(t^\alpha)$.

□

Def: $T(\alpha,\beta) := |\Omega_\beta(t^\alpha)|/|H(t^\alpha)|$

Theorem 1 *If α and β are partitions of n, and S_α and $S_{\beta'}$ are Young subgroups of S_n which correspond to α and β' then $i(IS_\alpha \uparrow S_n, AS_{\beta'} \uparrow S_n) = T(\alpha,\beta')$.*

Proof: By Lemma 2, $H(t^\alpha)$ acts in $\Omega_{\beta'}(t^\alpha)$ by left multiplication. By the counting formula, if $t^{\beta'} \in \Omega_{\beta'}(t^\alpha)$,as the stabilizer in $H(t^\alpha)$ of $t^{\beta'}$ is $\{Id\}$, the order of $t^{\beta'}$ is $|H(t^\alpha)|$. Let $\{t_1^{\beta'}, t_2^{\beta'}, \ldots, t_{T(\alpha,\beta)}^{\beta'}\}$ be a set of representatives of this orbits. Then, if $\gamma_s := \kappa(\{t_s^{\beta'}\},\{t^\alpha\})$, the formula

$$\phi_s(f_\beta)(\sigma_1\{t^\alpha\}) = (\text{sgn}\sigma_1)\gamma_s \left(\sum_{\sigma \in H(t^\alpha)\cap A_n} f_\beta(\sigma_1\sigma\{t_{i_s}^{\beta'}\}) - \sum_{\sigma \in (H(t^\alpha)\cap A_n^c)} f_\beta(\sigma_1\sigma\{t_{i_s}^{\beta'}\}\}) \right)$$

$\forall \sigma_1 \in S_n, \forall s \in \{i_1, i_2, \ldots, i_{T(\alpha,\beta)}\}$ defines an intertwinig operator between $A\ell(P_{\beta'})$ and $I\ell(P_\alpha)$. We will see that $\{\phi_s\}_{s=1}^{T(\alpha,\beta)}$ is a basis for the space of operators: There exist $\sigma_1 \in S_n$ and there exist $\sigma, \sigma' \in H(t^\alpha)$ so that $\sigma_1\sigma\{t_i^{\beta'}\} = \sigma_1\sigma'\{t_j^{\beta'}\}$ if and only if $\{t_i^{\beta'}\} = \sigma^{-1}\sigma'\{t_j^{\beta'}\}$ if and only if the orbit of $\{t_i^{\beta'}\}$ is the same as the orbit of

$\{t_j^{\beta'}\}$ under the action of $H(t^\alpha)$, this is, $i = j$ So we have proved that $\{\phi_s\}_{s=1}^{T(\alpha,\beta)}$ is a linearly independent set.

We will now see that it spans the space. By (1) we see that the support of an intertwining operator in which $\gamma_s \neq 0$ is at least the same as in ϕ_s, and because of Lemma 1 , the maximum support of ϕ is $\Omega_{\beta'}$, so the degrees of freedom are exactly $T(\alpha, \beta')$.

□

$\alpha \trianglelefteq \beta$ is defined in [1], pp 23 It is straightforward to notice that the following lemma is equivalent to lemma 1.4.20, proved in [1] pp. 28.

Lemma 3 *If $\alpha, \beta \vdash n$ and t^α is an α-tableau, then $\alpha \trianglelefteq \beta$ if and only if there exists a β'-tableau $t^{\beta'}$ such that any two points which occur in t^α in the same row occur in $t^{\beta'}$ in different rows.*

Using this Lemma and Theorem 1 we prove:

Corollary 1 *(Ruch-Schönhofer Theorem) If $\alpha, \beta \vdash n, S_\alpha$ and $S_{\beta'}$ are Young subgroups of S_n which correspond to α and β' , then $i(IS_\alpha \uparrow S_n, AS_{\beta'} \uparrow S_n) \neq 0$ if and only if $\alpha \trianglelefteq \beta$*

Our goal now is to prove: $i(IS_\alpha \uparrow S_n, AS_{\alpha'} \uparrow S_n) = 1$

Def: The set consisting of the elements belonging to the i^{th} row of $\rho \in P_\alpha$ will be called ρ_i.

Lemma 4 *Let $\rho \in P_\alpha$. Then $\pi \in \Omega_\beta(\rho)$ if and only if every $\rho_i \cap \pi_j$ has at most one element.*

Proof: ($\Longrightarrow$)If $a, b \in \rho_i \cap \pi_j$, the transposition (a, b) fixes π and ρ. In particular $(a, b) \in H(\rho) \setminus \{Id\}$.

($\Longleftarrow$) Let's suppose that every $|\rho_i \cap \pi_j| \leqslant 1$. If $\sigma \in H(\rho) \setminus \{Id\}$, there exists $x \in \{1, 2, \ldots, n\}$ such that $\sigma(x) \neq x$. Let j be such that $x \in \pi_j$, then $\sigma(x) \in \pi_j$, but then, if $x \in \rho_i$, we have $\sigma(x) \notin \rho_i$. As $\sigma(x) \in \sigma(\rho_i)$,it follows $\sigma(\rho_i) \neq \rho_i$, and so $\sigma\pi \neq \pi$. □

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_h)$ and $\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_k)$. We now define π' as the α'-tableau in which $\pi'_i := \{i, \alpha_1 + i, \alpha_1 + \alpha_2 + i, \ldots, \alpha_1 + \cdots \alpha_{\alpha'_1 - 1} + i\}$, and we define ρ' as the α-tableau in which $\rho'_i := \{\sum_{j=1}^{i-1} \alpha_j + 1, \sum_{j=1}^{i-1} \alpha_j + 2, \ldots, \sum_{j=1}^{i} \alpha_j\}$.Clearly π' is ρ' transposed.By Lemma 4 we see that $\pi' \in \Omega_{\alpha'}(\rho')$

Lemma 5 *$\pi \in \Omega_{\alpha'}(\rho')$ if and only if $\pi \in H(\rho')\pi'$*

Proof: ($\Longleftarrow$) Lemma 2.

($\Longrightarrow$) As a consecuence of Lemma 4, $|\rho'_i \cap \pi_j| \leqslant 1 \quad \forall i, j$.

We conclude that there exists $\prod_{i=1}^m \alpha'_i$ posibilities for ρ'_i, there exist $\prod_{\alpha'_i \geqslant 2} (\alpha'_i - 1)$ posibilities for ρ'_2, etc. Then $|\Omega_{\alpha'}(\rho')| = \prod_{i=1}^m \alpha'_i!$ but $|H(\rho')\pi'| = |H(\rho'))| = \prod_{i=1}^m \alpha'_i!$, then $H(\rho')\pi' = \Omega_{\alpha'}(\rho')$. □.

Lemma 5 and Theorem 1 demonstrate that $i(IS_\alpha \uparrow S_n, AS_{\alpha'} \uparrow S_n) = 1$. So we wil call $[\alpha] := IS_\alpha \uparrow S_n \cap AS_{\alpha'} \uparrow S_n$. Clearly $[\alpha]$ is an irreducible representation. From Theorem 1 we deduce that ϕ is uniquely determined up to scalar, and $[\alpha] = Im(\phi) \subset I\ell(P_\alpha)$. We will now prove that there are no other irreducible representation of S_n

Theorem 2 $\{[\alpha] | \alpha \vdash n\}$ *is the complete set of equivalence classes of ordinary irreducible representations of* S_n

Proof: We need only to show that $[\alpha] = [\beta]$ implies $\alpha = \beta$, for then the cardinality of $\{[\alpha] | \alpha \vdash n\}$ is equal to the number of conjugacy classes of S_n , so that this system must be complete. But if $[\alpha] = [\beta]$,

$$i(IS_\alpha \uparrow S_n, [\beta]) = i(IS_\alpha \uparrow S_n, [\alpha]) = 1 = i(IS_\beta \uparrow S_n, [\beta]) = i(IS_\beta \uparrow S_n, [\alpha])$$

Thus by Corollary 1 we have both $\alpha \trianglelefteq \beta$ and $\beta \trianglelefteq \alpha$ which imply $\alpha = \beta$.

References

[1] J. Gordon, A. Kerber: *The representation theory of the symmetric group*, Encyclopedia of mathematics and its applications, vol. 16, pp. 15-38,1981.

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